Conjugate points

Let $\gamma : [0,a] \to M$ be a geodesic.
The point $\gamma(t_0)$ is <u>conjugate</u> to $\gamma(0)$ along γ, $t_0 \in (0,a]$, iff there exists a Jacobi field J along γ, not identically equal to 0, with $J(0) = 0 = J(t_0)$. The moksimum number of such linearly independent Jacobi fields is called the <u>multiplicity</u> of the conjugate point $\gamma(t_0)$.

Corollary

If $\dim M = n$, the multiplicity of conjugate points never exceed number $n-1$.

Proof

Demanding $J(0) \overset{!}{=} 0$ we have n independent Jacobi fields $J_1, \dots J_n$ by setting initial condition for the first derivative $J_1'(0), J_2'(0), \dots, J_n'(0)$ to be linearly independent.

Among these n independent solutions there is $J(t) = t\gamma'(t)$ which satisfies $J(0) = 0$ but which never vanishes for $t \neq 0$

So we have <u>at most</u> $n-1$ independent solutions that may satisfy $J(0) = 0 = J(t_0)$ for some $t_0 \neq 0$ $\square$

Example

$\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} ; |x|^2 = 1 \}$

$\Rightarrow$ space of constant curvature with $K = 1$.

$\Rightarrow \quad J(t) = (\sin t)\, w(t)$ where $w(t)$ is $\perp$ to geodesic (grate circle)

Conjugate point to $t=0$ is $t_0 = \pi$, which means that conjugate points are antipodal on $\mathbb{S}^n$.

Since on n-dimensional sphere we hove $n-1$ linearly indep. vectors orthogonal to a given one (e.g. to the tangent vector to a geodesic) we see that on the sphere conjugate points has maximal multiplicity $= n-1$.

Conjugate locus

The set of first conjugate points to the point $p \in M$, for all the geodesics that start at p is called <u>conjugate locus of p</u>. We denote it by $C(p)$.

<u>Example</u> on $\mathbb{S}^n$ $\quad C(p) =$ antipodal point to p on $\mathbb{S}^n$.

Usually, however, $C(p)$ is a curve on M with singular points on it. (See DoCarmo, p. 270, figure 4, for example)

Proposition

1) $q = \gamma(t_0)$ is a conjugate point to $p = \gamma(0)$ along a geodesic γ if and only if $v_0 = t_0\gamma'(0)$ is a critical point for $\exp_p$.

2) Moreover, the multiplicity of q is equal to the dimension of kernel of $(d\exp_p)_{v_0}$.

Proof

Ad 1) $J(0) = 0 = J(t_0)$. We set $v = \gamma'(0)$, $\omega = J'(0)$. Then

$$J(t) = (d\exp_p)_{tv}(t\omega).$$

If $\omega \neq 0$, what we assume, $J(t)$ is nonzero vector field along γ.

$$\Rightarrow 0 = J(t_0) = (d\exp_p)_{t_0 v}(t_0\omega) \iff (d\exp_p)_{t_0 v} = 0$$

$\Rightarrow$ $t_0 v$ is critical point for $\exp_p$.

Ad 2) If we have k linearly independent $t_0\omega$'s s.t.

$(d\exp_p)_{t_0 v}(t_0\omega) = 0$ $\Rightarrow$ we can use each of them to get J s.t. $J(0) = 0 = J(t_0)$. In this way we obtain k linearly independent J's.

$\square$.

Prop

$$\left(\begin{array}{c} J \text{ - Jacobi field along } \gamma \\ \gamma \text{ - geodesic} \\ [a,a] \ni t \longrightarrow \gamma(t) \end{array}\right) \Rightarrow \left(\begin{array}{l} g(J(t), \gamma'(t)) = \\ = g(J'(0), \gamma'(0))\,t + g(J(0), \gamma'(0)) \\ \qquad\qquad \forall\, t \in [a,a] \end{array}\right)$$

Proof

$J'' = R(\gamma', J)\gamma'$. Note: $'$ on tensors mean $\nabla_{\gamma'}$. In particular on vectors it means $\frac{D}{dt}$.

$$g(J',\gamma')' = g(J'',\gamma') + \cancel{g(J', \tfrac{D\gamma'}{dt})} = g(R(\gamma',J)\gamma',\gamma') \underset{\substack{\uparrow \\ \text{antisymmetry} \\ \text{in } (\dots \gamma', \gamma')}}{=} 0$$

$$\Rightarrow \quad \underline{g(J'(t),\gamma'(t)) = g(J'(0),\gamma'(0))}$$

In addition:

$$\boxed{g(J,\gamma')' = g(J',\gamma') = g(J'(0),\gamma'(0))}$$

differential equation to $g(J,\gamma')$

$$\Rightarrow \quad g(J,\gamma') = g(J'(0),\gamma'(0))\,t + \underset{\substack{\shortparallel \\ g(J(0),\gamma'(0))}}{\text{const}}.$$

$\square$

In particular:

$$\boxed{\begin{array}{ll} \text{if} & g(J,\gamma')(t_1) = g(J,\gamma')(t_2) \\ \Rightarrow & g(J,\gamma') = \text{const.} \end{array}} \Rightarrow \boxed{\begin{array}{c} J(0) = J(a) = 0 \\ \Downarrow \\ g(J,\gamma') \equiv 0 \end{array}}$$

if a geodesic admit conjugate points $\Rightarrow$ the Jacobi field for which $J(0) = 0 = J(t_0)$ is always orthogonal to γ.

Proposition

$\gamma : [0,a] \to M$ geodesic
$V_1 \in T_{\gamma(0)}M$, $V_2 \in T_{\gamma(a)}M$
$\gamma(a)$ is not conjugate to $\gamma(0)$
$\Longrightarrow$ there exists a unique Jacobi field along γ s.t. $J(0) = V_1$, $J(a) = V_2$.

Proof

Let $\mathcal{J}$ be the space of Jacobi fields J with $J(0)=0$.

Define $\Theta : \mathcal{J} \to T_{\gamma(a)}M$ by

$\Theta(J) = J(a)$ $\quad J \in \mathcal{J}$

then Θ is injective

Since $\gamma(a)$ is not a conjugate point to $\gamma(0)$ (because $J(a) \neq 0$ for all $J \in \mathcal{J}$ s.t. $J \neq 0$ and Θ is linear).

$\dim \mathcal{J} = n = \dim T_{\gamma(a)}M \Rightarrow \Theta$ is an isomorphism.

$\Rightarrow$ given $V_2 \in T_{\gamma(a)}M$ ~~there~~ exists unique $\bar{J}_1$ s.t. $\bar{J}_1(0)=0$, $\bar{J}_1(a) = V_2$.

Reversing the argument, i.e. starting with a there exists unique $\bar{J}_2$ s.t. $\bar{J}_2(a) = 0$, $\bar{J}_2(0) = V_1$

take $J = \bar{J}_1 + \bar{J}_2$

$\square$.

SO(1,2)

$$E_1 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0&1&0\\ -1&0&0\\ 0&0&0 \end{pmatrix}$$

$$[E_1 E_2] = E_3, \quad [E_3 E_1] = E_2, \quad [E_2 E_3] = E_1$$

$$g = \exp(t_1 E_1)\exp(t_2 E_2)\exp(t_3 E_3)$$

$$\vartheta_{MC} = g^{-1}dg =$$

$$\overbrace{(dt_1 - \sinh t_2\, dt_3)}^{A_1} E_1 + \overbrace{(\cosh t_1\, dt_2 + \cosh t_2 \sinh t_1\, dt_3)}^{A_2} E_2$$

$$+ \overbrace{(\cosh t_1 \cosh t_2\, dt_3 + \sinh t_1\, dt_2)}^{A_3} E_3$$

$$A_3^2 - A_2^2 = \cosh^2 t_2\, dt_3^2 - dt_2^2$$

$$dA_1 = -A_2 A_3$$
$$dA_2 = A_1 A_3$$
$$dA_3 = A_1 A_2$$

$$\theta^1 = A_3, \quad \theta^2 = A_2 \qquad g = \theta^{12} - \theta^{22}$$

$$d\theta^1 = A_1\theta^2 = -\Gamma^1{}_2\theta^2 = -\Gamma_{12}\theta^2$$
$$d\theta^2 = A_1\theta^1 = -\Gamma^2{}_1\theta^1 = \Gamma_{21}\theta^1 = -\Gamma_{12}\theta^1 \qquad \Rightarrow \quad -\Gamma_{12} = A_1$$

$$\Omega_{12} = d\Gamma_{12} = -dA_1 = A_2 A_3 = \theta^2\theta^1 = -\theta^1 \wedge \theta^2$$

$$\underline{K = -1\ !}$$

But also

$$g = A_1^2 + A_2^2 \qquad \text{is such that} \qquad \mathcal{L}_{X_3} g = 0$$

$(X_3\ X_2\ X_1)$ dual to $A_3\ A_2\ A_1$

integrate X_3!